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ON THE ANALYTIC CONTINUATION OF THE MINAKSHISUNDARAM-PLEIJEL ZETA FUNCTION FOR COMPACT RIEMANN SURFACES

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ABSTRACT. A formula is derived for the Minakshisundaram-Pleijel zeta function in the half-plane $\operatorname{Re} s < 0$.

Let S be a compact Riemann surface, which we will regard as the quotient of the upper half-plane H by a discontinuous group Γ of hyperbolic transformations. We will assume that H is endowed with the metric $y^{-2}((dx)^2 + (dy)^2)$, and we will denote the area of S by A. Let $0 = \lambda_0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots$ be the eigenvalues corresponding to the problem $\Delta f + \lambda f = 0$ on S, where Δ is the Laplace operator on S, derived from the metric induced on S by that of H. In the coordinates of H, the Laplacian is $y^2(\partial^2/\partial x^2 + \partial^2/\partial y^2)$. Finally, let $Z(s) = \Sigma_{n=1}^{\infty} \lambda_n^{-s}$. Since it is known [3] that $N(T) = \Sigma_{\lambda_n < T} 1$ is asymptotic to $(A/4\pi)T$, it follows that the series for $Z(s)$ converges absolutely in the half-plane $\operatorname{Re} s > 1$.

In this note we will use the Selberg trace formula to derive an expression for the continuation of $Z(s)$ in the half-plane $\operatorname{Re} s < 0$. Accounts of the trace formula can be found in [1], [2], and [4]. The formula, adjusted to the present situation, goes as follows.

Suppose $h(z)$ is an even function, holomorphic in a strip of the form $|\operatorname{Im} z| < \frac{1}{2} + \epsilon$ $(\epsilon > 0)$, and satisfying a growth condition of the form $|h(z)| = O((1 + |z|^2)^{-1-\epsilon})$ uniformly in the strip. Associate with the sequence $\lambda_0, \lambda_1, \lambda_2, \cdots$ of eigenvalues the set R consisting of those numbers which satisfy an equation of the form $\lambda_n = \frac{1}{4} + r^2$ $(n = 0, 1, 2, \cdots)$. Apart from the possibility $r = 0$, the elements of R will then occur in pairs, of which each member is the negative of the other, and it is always the case that every element of R is either real or pure imaginary, with imaginary part $\leqslant \frac{1}{2}$. If one of the λ_n's happens to be $\frac{1}{4}$, the corresponding $r = 0$ will be counted with double multiplicity in its occurrence on the left side of the trace formula.

Now all the elements of Γ except the identity are hyperbolic. I.e., each $\gamma \in \Gamma$ is conjugate in $PSL(2, R)$ to a unique transformation of the form

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$z \longrightarrow e^{l_\gamma} z$, where l_γ is real and positive. For geometric reasons, we will call the number l_γ the length of the transformation γ (cf. [2]). Clearly l_γ is the same within a conjugacy class. We will denote by $\{\gamma\}$ the conjugacy class corresponding to γ within Γ itself. Also, we will call $\gamma \in \Gamma$ primitive, if it is not a positive integral power of any other element of Γ. Clearly we can speak of a conjugacy class in Γ as being primitive. The trace formula then reads

$$\sum_{r_n \in R} h(r_n) = \frac{A}{2\pi} \int_{-\infty}^{\infty} h(r)\, r \tanh \pi r\, dr + \sum_{\{\gamma\}_p} \sum_{n=1}^{\infty} (l_\gamma \operatorname{csch} \tfrac{1}{2} n l_\gamma) \hat{h}(n l_\gamma),$$

where

$$\hat{h}(x) = (2\pi)^{-1} \int_{-\infty}^{\infty} e^{-ixr} h(r)\, dr,$$

and the outer sum is taken over all primitive conjugacy classes in Γ. Moreover, all series in the formula converge absolutely.

In order to study $Z(s)$, it is convenient to begin by studying a more general Dirichlet series. Namely, suppose $\epsilon \geqslant 0$, and define $Z_\epsilon(s) = \Sigma_{n=0}^\infty (\lambda_n + \epsilon)^{-s}$. As before, the series converges absolutely in the half-plane $\operatorname{Re} s > 1$. Next define $\alpha \geqslant \tfrac{1}{2}$ by requiring that $\alpha^2 - \tfrac{1}{4} = \epsilon$. Letting t be a positive number and setting $h(r) = e^{-(\alpha^2 + r^2)t}$ in the trace formula, we obtain

$$\sum_{n=0}^{\infty} e^{-(\lambda_n + \epsilon)t} = \frac{A}{4\pi} \int_{-\infty}^{\infty} e^{-(\alpha^2 + r^2)t}\, r \tanh \pi r\, dr \tag{1}$$
$$+ \frac{1}{2}(4\pi t)^{-1/2} \sum_{\{\gamma\}_p} \sum_{n=1}^{\infty} (l_\gamma \operatorname{csch} \tfrac{1}{2} n l_\gamma) e^{-(4\alpha^2 t^2 + n^2 l_\gamma^2)/4t}$$

with all series convergent.

Denote by $\theta_1^\epsilon(t)$ and $\theta_2^\epsilon(t)$, respectively, the first and second terms on the right side of (1). Then $\Sigma_{n=1}^\infty e^{-(\lambda_n + \epsilon)t} = \theta_1^\epsilon(t) + \theta_2^\epsilon(t) - e^{-\epsilon t}$.

Throughout what follows, when we say that a result holds uniformly in ϵ, we will mean that it holds uniformly for $\epsilon \in [0, 1]$, or what is the same thing, for $\alpha \in [1/2, \sqrt{5}/2]$.

The following two lemmas are obvious.

LEMMA 1. $\Sigma_{n=1}^\infty e^{-(\lambda_n + \epsilon)t} = O(e^{-\lambda_1 t})$ *uniformly in* ϵ, *as* $t \longrightarrow \infty$.

LEMMA 2. $\theta_1^\epsilon(t) = O(e^{-t/4})$ *uniformly in* ϵ, *as* $t \longrightarrow \infty$.

Combining Lemmas 1 and 2, we obtain the following lemma.

LEMMA 3. $\theta_2^\epsilon(t) - e^{-\epsilon t} = O(e^{-(\min(\lambda_1, 1/4))t})$ *uniformly in* ϵ, *as* $t \longrightarrow \infty$.

LEMMA 4. $\theta_2^\epsilon(t)$ *is of rapid decrease as* $t \longrightarrow 0$, *uniformly in* ϵ. *I.e., for any negative integer* k, $t^k \theta_2^\epsilon(t) \longrightarrow 0$ *as* $t \longrightarrow 0$, *uniformly in* ϵ.

PROOF. For any k, $t^k\theta_2^\epsilon(t)$ is equal to a convergent series of positive terms. Moreover, for t less than some η, which depends on k but can be taken independent of ϵ, the terms all tend monotonically to zero as $t \downarrow 0$. The result then follows from Beppo Levi's theorem.

LEMMA 5. *Define* $\theta_2^\epsilon(0) = 0$. *Then for each* $\epsilon \geqslant 0$, $\theta_2^\epsilon(t)$ *is continuous, and the series for* $\theta_2^\epsilon(t)$ *converges uniformly to* $\theta_2^\epsilon(t)$ *on compact subsets of* $[0, \infty)$.

PROOF. For any $\epsilon \geqslant 0, \theta_2^\epsilon(t)$ is continuous at $t = 0$ by Lemma 4. For $t > 0, \theta_2^\epsilon(t)$ is clearly continuous, being a linear combination of three continuous functions. Moreover, since all the terms of the series that defines $\theta_2^\epsilon(t)$ are positive, it follows from Dini's theorem that the series converges uniformly on compact subsets of $[0, \infty)$.

LEMMA 6. *On any compact subset of* $[0, \infty), \theta_2^\epsilon(t) \longrightarrow \theta_2^0(t)$ *uniformly, as* $\epsilon \longrightarrow 0$.

PROOF. $\theta_2^\epsilon(t)$ increases monotonically to $\theta_2^0(t)$ as $\epsilon \downarrow 0$. The result thus follows from Dini's theorem.

Now suppose $\operatorname{Re} s > 1$. Taking the Mellin transform of $\sum_{n=1}^\infty e^{-(\lambda_n+\epsilon)t}$, and adding $1/s$ to the result, we obtain

$$\Gamma(s)Z_\epsilon(s) + \frac{1}{s} = \frac{A\Gamma(s)}{4\pi}\int_{-\infty}^\infty (\alpha^2 + r^2)^{-s}\, r \tanh \pi r\, dr$$
$$+\int_0^\infty (\theta_2^\epsilon(t) - e^{-\epsilon t})t^s \frac{dt}{t} + \frac{1}{s},$$

or

$$\begin{aligned}\Gamma(s)Z_\epsilon(s) + \frac{1}{s} &= \frac{A\Gamma(s)}{8(s-1)}\int_{-\infty}^\infty (\alpha^2 + r^2)^{1-s}\operatorname{sech}^2 \pi r\, dr \\ &+\int_0^1 (\theta_2^\epsilon(t) + 1 - e^{-\epsilon t})t^s \frac{dt}{t} + \int_1^\infty (\theta_2^\epsilon(t) - e^{-\epsilon t})t^s \frac{dt}{t}.\end{aligned} \tag{2}$$

In view of Lemmas 3 and 4, the right side of the last equation gives, for any $\epsilon \geqslant 0$, a meromorphic continuation of $\Gamma(s)Z_\epsilon(s) + 1/s$, and hence of $\Gamma(s)Z_\epsilon(s)$, into $\operatorname{Re} s > -1$, and indeed, into the whole plane if $\epsilon = 0$ (since the first and third integrals are entire for any $\epsilon \geqslant 0$, and the second integral is holomorphic in $\operatorname{Re} s > -1$, and entire if $\epsilon = 0$). If $\epsilon > 0$, and we observe, using the power series for $e^{-\epsilon t}$, that $\int_0^1 (1 - e^{-\epsilon t})t^s\, dt/t$ can be continued to the left of $\operatorname{Re} s > -1$, with simple poles at the negative integers, we obtain a meromorphic continuation of $\Gamma(s)Z_\epsilon(s)$ into the entire plane in this case as well. Since it is clear from this that the only possible poles of $\Gamma(s)Z_\epsilon(s)$ are simple poles at $1, 0, -1, -2, \cdots$, with the pole at $s = 1$ always present, we conclude that for any $\epsilon \geqslant 0$, $Z_\epsilon(s)$ can be meromorphically continued into the whole plane, with a single simple pole at $s = 1$, having residue $A/4\pi$ (since $\int_{-\infty}^{\infty} \operatorname{sech}^2 \pi r\, dr = 2/\pi$).

Suppose now $\operatorname{Re} s > -1$, and $s \neq 0, 1$. Then in view of Lemma 6, it is evident, by inspecting the right side of (2), bearing in mind Lemmas 3 and 4, that $\Gamma(s)Z_\epsilon(s) + 1/s \longrightarrow \Gamma(s)Z(s) + 1/s$, and hence $\Gamma(s)Z_\epsilon(s) \longrightarrow \Gamma(s)Z(s)$, as $\epsilon \longrightarrow 0$.

On the other hand, if $\operatorname{Re} s > 0$,

$$\Gamma(s)Z_\epsilon(s) = \frac{A\Gamma(s)}{8(s-1)}\int_{-\infty}^{\infty}(\alpha^2 + r^2)^{1-s}\operatorname{sech}^2 \pi r\, dr + \int_0^\infty (\theta_2^\epsilon(t) - e^{-\epsilon t})t^s\,\frac{dt}{t},$$

and if $\epsilon > 0$, it is permissible to split the last integral into two integrals, since it follows from Lemma 3 that for positive ϵ, $\theta_2^\epsilon(t)$ is of exponential decrease as $t \longrightarrow \infty$. We thus obtain, at first for $\operatorname{Re} s > 0$, and then for the whole plane by analytic continuation,

$$\Gamma(s)Z_\epsilon(s) = \frac{A\Gamma(s)}{8(s-1)}\int_{-\infty}^{\infty}(\alpha^2 + r^2)^{1-s}\operatorname{sech}^2 \pi r\, dr + \int_0^\infty \theta_2^\epsilon(t)\, t^s\,\frac{dt}{t} - \Gamma(s)\,\epsilon^{-s}.$$

Now taking $-1 < \operatorname{Re} s < 0$, letting $\epsilon \longrightarrow 0$, and bearing in mind that by Lemma 3, $\theta_2^\epsilon(t) = O(1)$ uniformly in ϵ, as $t \longrightarrow \infty$, we find that

$$\Gamma(s)Z(s) = \frac{A\Gamma(s)}{8(s-1)}\int_{-\infty}^{\infty}(\tfrac{1}{4} + r^2)^{1-s}\operatorname{sech}^2 \pi r\, dr + \int_0^\infty \theta_2^0(t)\, t^s\,\frac{dt}{t}.$$

Since both integrals are holomorphic in $\operatorname{Re} s < 0$, we have obtained an expression for $\Gamma(s)Z(s)$ in the left half-plane.

Let us examine $\int_0^\infty \theta_2^0(t)\, t^s\, dt/t$, assuming $\operatorname{Re} s < 0$. Now

$$\int_0^\infty e^{-(t^2+(nl_\gamma)^2)/4t}\, t^{s-1/2}\,\frac{dt}{t} = 2(nl_\gamma)^{s-1/2}K_{1/2-s}(\tfrac{1}{2}nl_\gamma) \qquad [5, \text{p. } 183],$$

so we obtain the following result (the interchange of summation and integration being justified by Lemma 5):

THEOREM 1. *If* $\operatorname{Re} s < 0$,

$$\Gamma(s)Z(s) = \frac{A\Gamma(s)}{8(s-1)}\int_{-\infty}^{\infty}(\tfrac{1}{4} + r^2)^{1-s}\operatorname{sech}^2 \pi r\, dr + (4\pi)^{-1/2}\sum_{\{\gamma\}_p}\sum_{n=1}^{\infty}(l_\gamma/n)^{1/2}(\operatorname{csch} \tfrac{1}{2}nl_\gamma)(nl_\gamma)^s K_{1/2-s}(\tfrac{1}{2}nl_\gamma).$$

Now if $\operatorname{Re} s < \tfrac{1}{2}$,

$$(nl_\gamma)^s K_{1/2-s}(\tfrac{1}{2}nl_\gamma) = \pi^{-1/2}\,\Gamma(1-s)\,(nl_\gamma)^{1/2}\int_0^\infty ((\tfrac{1}{2}nl_\gamma)^2 + x^2)^{-(1-s)}\cos x\,dx$$

[5, p. 172],

so if we define $\phi_s(A)$, for $\operatorname{Re} s > \frac{1}{2}$ and positive A, by setting $\phi_s(A) = (2\pi)^{-1}\int_0^\infty ((A/2)^2 + x^2)^{-s}\cos x\,dx$, we obtain the following reformulation of Theorem 1.

THEOREM 2. *If* $\operatorname{Re} s < 0$,

$$(3)\qquad \Gamma(s)Z(s) = \frac{A\Gamma(s)}{8(s-1)}\int_{-\infty}^\infty (\tfrac{1}{4} + r^2)^{1-s}\operatorname{sech}^2 \pi r\,dr + \Gamma(1-s)\sum_{\{\gamma\}_p}\sum_{n=1}^\infty l_\gamma(\operatorname{csch}\tfrac{1}{2}nl_\gamma)\,\phi_{1-s}(nl_\gamma).$$

Suppose now $\operatorname{Re} s > 1$. Then

$$\frac{A\Gamma(s)}{8(s-1)}\int_{-\infty}^\infty (\tfrac{1}{4} + r^2)^{1-s}\operatorname{sech}^2\pi r\,dr = \frac{A\Gamma(s)}{2\pi}\int_0^\infty (\tfrac{1}{4} + r^2)^{-s}\,r\tanh \pi r\,dr.$$

Now as we have pointed out, it is well known that $\Sigma_{\lambda_n < T}\, 1 \sim AT/4\pi$, so it follows that $\Sigma_{0<r_n<T}\, 1 \sim AT^2/4\pi$. But $(A/2\pi)\int_0^T r\tanh\pi r\,dr \sim AT^2/4\pi$, and in view of the trace formula, is the correct principal term in the asymptotic analysis of $\Sigma_{0<r_n<T}\, 1$. This suggests defining a remainder term

$$R(T) = \sum_{0<r_n<T} 1 - \frac{A}{2\pi}\int_0^T r\tanh\pi r\ dr.$$

Then if we denote the eigenvalues in $(0, \frac{1}{4}]$ by $\lambda_1, \cdots, \lambda_N$, and define $\lambda(r) = \frac{1}{4} + r^2$, Theorem 2 and the previous arguments tell us that $\Gamma(s)\,\{\Sigma_{n=1}^N \lambda_n^{-s} + \int_0^\infty (\lambda(r))^{-s}\,dR(r)\}$ can be meromorphically continued from $\operatorname{Re} s > 1$ into the whole plane, and for $\operatorname{Re} s < 0$, equals $\Gamma(1-s)\,\Phi(1-s)$, where $\Phi(s)$ is defined in the half-plane $\operatorname{Re} s > 1$ by setting

$$\Phi(s) = \sum_{\{\gamma\}_p}\sum_{n=1}^\infty l_\gamma(\operatorname{csch}\tfrac{1}{2}nl_\gamma)\,\phi_s(nl_\gamma).$$

If, now, we define $R^*(T) = R(\sqrt{T - \frac{1}{4}})$, integrate by parts, and make the change of variable $\lambda = \lambda(r)$, the previous statement becomes the statement that $\Gamma(s)\,\{\Sigma_{n=1}^N \lambda_n^{-s} + s\int_{1/4}^\infty \lambda^{-s-1}R^*(\lambda)\,d\lambda\}$ can be meromorphically continued into the whole plane, and for $\operatorname{Re} s < 0$, equals $\Gamma(1-s)\,\Phi(1-s)$.

Thus, setting

$$\Psi(s) = s\int_{1/4}^\infty \lambda^{-s-1}R^*(\lambda)\,d\lambda = s\int_{\log 1/4}^\infty e^{-\lambda s}R^*(e^\lambda)\,d\lambda = \int_{\log 1/4}^\infty e^{-\lambda s}\,dR^*(e^\lambda),$$

we find that $\Psi(s)$, the Laplace transform of the exponential form of the eigenvalue remainder measure, satisfies the following identity:

THEOREM 3. *If* $\operatorname{Re} s < 0$, $\Gamma(s)\{\sum_{n=1}^{N} \lambda_n^{-s} + \Psi(s)\} = \Gamma(1-s)\Phi(1-s)$.

COROLLARY. *If there are no eigenvalues in* $(0, 1/4]$ *and* $\operatorname{Re} s < 0$, *we have* $\Gamma(s)\Psi(s) = \Gamma(1-s)\Phi(1-s)$.

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